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NOTE ON POLES AND POLARS.

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While the matter here presented is not new to the mathematician, the method appears, from a pedagogical point of view, to have some advantages.

Through the point $O\equiv(x', y')$ a straight line is drawn cutting a conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ at A and B.

Let C be a point in AB such that $OC=\dfrac{2OA\times OB}{OA+OB}$.

We propose to find the locus of C, when the straight line turns about the point O.

Transferring the origin to (x', y') the new equation is

$$ax^2+2hxy+by^2+2(ax'+hy'+g)+2(hx'+by'+f)+$$
$$ax'^2+2hx'y'+by'^2+2gx'+2fy'+c=0.$$

For the sake of brevity we write this

$$ax^2+2hx'y'+by'^2+2g'x'+2f'y'+c'=0,$$

where $g'=ax'+hy'+g$, $f'=hx'+by'+f$, $c'=ax'^2+2hx'y'+$ etc.

Putting $x=r\cos\theta$, $y=r\sin\theta$, and arranging according to powers of r,

$$(a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta)r^2+2(g'\cos\theta+f'\sin\theta)+c'=0.$$

Then $OC = \frac{2 \text{ product of roots}}{\text{sum of roots}} = \frac{-c'}{g'\cos\theta + f'\sin\theta}$.

Putting $OC = \rho$, we have $g'\rho\cos\theta + f'\rho\sin\theta + c' = 0$.

But $\rho\cos\theta = x$, $\rho\sin\theta = y$; hence $g'x + f'y + c' = 0$.

This straight line is the polar of the point (0, 0) with respect to the conic $ax^2 + 2hxy + by^2 + 2g'x + 2f'y + c' = 0$.

Transferring to the old origin the equation of the polar becomes

$$(ax' + hy' + g)(x - x') + (hx' + by' + f)(y - y') + ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c = 0,$$

which may be written

$$axx' + h(xy' + x'y) + bgy' + g(x + x') + f(y + y') + c = 0.$$

This is the equation of the polar of $x'y'$ with respect to the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

If the point C be fixed then will O describe the polar of C ; for, we have on putting $OC = r$, $CA = a$, $CB = b$,

$$r = \frac{2(r+a)(r+b)}{2r + a + b}.$$

Clearing and solving for r, we find $-r = \frac{2ab}{a+b}$, *i. e.* $CO = \frac{2CA \times CB}{CA + CB}$.

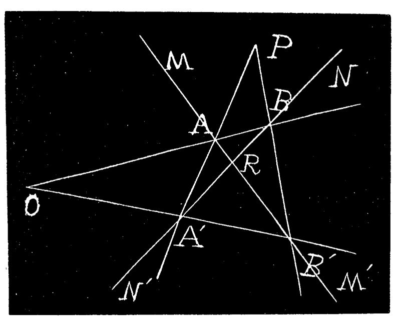
Hence: *If the point P lies on the polar of the point Q, then will Q lie on the polar of P.*

If the line OAB is drawn tangent to the curve the points A and B coincide with C. Hence:

The polar of a point passes through the points of contact of the tangents drawn through the point.

If O be taken on the curve, C will coincide with O, and the polar is the tangent at O.

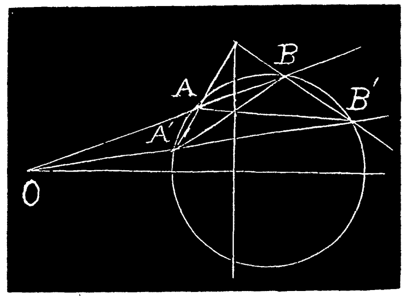
The polar of a point with respect to a pair of straight lines passes through the intersection of the lines ; for, on drawing a line through O and the point of intersection, A and B coincide with C.



To draw the polar of a point with respect to a pair of straight lines: Through O draw two secants cutting the lines at A and B, A' and B', respectively. Join AA', BB' and produce till they meet at P. Join P with common point of given lines R. PR is the polar. For, the polar of AA' and BB' passes through P and cuts AB and $A'B'$ at the same points as the polar with respect to NN' and MM'. The line OR is the polar of P with respect to either of the two pairs of

lines. OP is the polar of R with respect to the same lines ; in other words the triangle OPR is self-conjugate.

To draw the polar of a point with respect to a conic: Draw any two secants through O, join AB', $A'B$, AA', BB'. The polar with respect to the conic is the same as the polar with respect to AB' and $A'B$, or with respect to AA' BB'.



Construction of Pole and Polar when center is given: The tangents at points (x_1, y_1) (x_2, y_2) of the curve are

$$axx_1+h(xy_1+x_1y)+byy_1+g(x+x_1)+f(y+y_1)+c=0,$$

$$axx_2+h(xy_2+x_2y)+byy_2+g(x+x_2)+f(y+y_2)+c=0.$$

Subtraction gives

$$ax(x_1-x_2)+hx(y_1-y_2)+hy(x_1-x_2)+by(y_1-y_2)+g(x_1-x_2)+f(y_1-y_2)=0,$$

$$\text{or } (ax+hy+g)(x_1-x_2)+(hx+by+f)(y_1-y_2)=0,$$

the equation of a line through the intersection of the tangents. This line passes through the center, since the coördinates of the center cause $ax+hy+g$, $hx+by+f$, to vanish.

Putting $y_1-y_2=m(x_1-x_2)$, we may write this equation

$$(a+hm)x+(h+bm)y+(g+fm)=0.$$

This is the equation of the diameter which bisects the chord.

This may also be shown by putting $\frac{x_1+x_2}{2}$ for x, $\frac{y_1+y_2}{2}$ for y, the equation being satisfied thereby.

Finally: *The pole of a given line lies on the diameter which bisects the chord intercepted on that line by the conic.*

To find the pole of a given line : Draw the diameter conjugate to the line, join its extremities with any point of the line produced to cut the conic. Through the points of intersection draw a line to cut the diameter.

The construction in case of the parabola is easily effected by remembering that the center is at infinity.